Distribution of Drifting Bodies in the Ocean Current.

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When some of sea-weeds, drifting timbers, floating fish-eggs, planktons, fish larvae, etc. are put in the considerably strong oceanic current, they will be carried away just as is the case with the drift of current-bottles. It may be supposed that when thrown in the sea they will scatter gradually in the current-area during its drift with the elapse of time.

Assuming that the time of recovery of every bottle coincides with the time of arrival at its destination,[1] Fig. 1, which shows the relation between

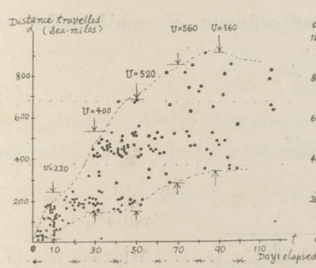

Fig. 1. a. Bottles thrown in Wakasa Bay. July 11, 1930.

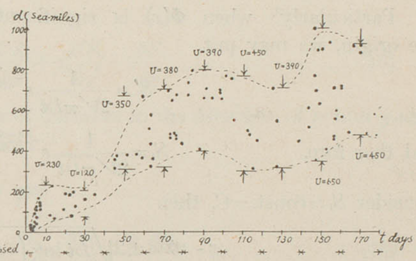

Fig. 1. b. Bottles thrown in Tusima Channel. Early June, 1932.

the days "out" and the distance travelled in sea-miles in three bottle-experiments in the Japan Sea, is considered to represent the scattering conditions of current-bottles during their drift. The scattering phenomena of these bottles may be correlated with the distributions of fish-eggs and fish larvae (e.g., sardine in the Tusima Current) or some species of planktons. In this paper it is tried to give a theoretical study on the results of current-bottle experiments with the purpose of future application to the biological problems above cited.

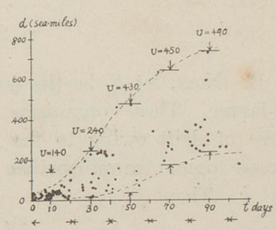

Fig. 1. c. Bottles thrown in Tusima Channel. Early Oct., 1933.

I ONE DIMENSIONAL DIFFUSION IN THE UNIFORM FLOW.

A. Let v be the velocity of flow of the medium in the direction of x-

1) Practically the time of recovery lags behind the time of arrival.

axis, then the equation of diffusion of substance, with concentration S_1, in the moving medium may be written as:

$$\frac{\partial S_1}{\partial t}=D\frac{\partial^2 S_1}{\partial x^2}-v\frac{\partial S_1}{\partial x}, \quad \text{......(1)}$$

where t denotes time, and D the coefficient of diffusion.

Putting
$$S_1=S'.e^{\frac{v}{2D}x-\frac{v^2t}{4D}} \quad \text{......(2)}$$

in (1), we get
$$\frac{\partial S'}{\partial t}=D\frac{\partial^2 S'}{\partial x^2}, \quad \text{......(3)}$$

of which the solution giving $S'=\Phi(x)$ at $t=0$, is

$$S'=\frac{1}{2\sqrt{\pi Dt}}\int_{-\infty}^{+\infty}\Phi(\alpha)e^{-\frac{(\alpha-x)^2}{4Dt}}d\alpha. \quad \text{......(4)}$$ [1]

Particularly[2] when $\Phi(x)$ is significant only in a small region about the origin, we may put

$$S'=\frac{1}{2\sqrt{\pi Dt}}e^{-\frac{x^2}{4Dt}}, \quad \text{......(5)}$$

and therefore,
$$S_1=\frac{1}{2\sqrt{\pi Dt}}e^{-\frac{(x-vt)^2}{4Dt}}. \quad \text{......(6)}$$

Consider $S_1=\text{const.}=C$, then

$$x-vt=\pm 2\sqrt{Dt\log\left(\frac{1}{2\sqrt{\pi Dt}C}\right)}. \quad \text{......(7)}$$

The time when $(x-vt)^2$ attains its maximum can be obtained from

$$2\sqrt{\pi Dt}C=\frac{1}{\sqrt{e}}. \quad \text{......(8)}$$

B. Next, let S_2 be the density of some species of plankton, fish-eggs, or fish-larvae. These organisms may decrease gradually in number in the course of drift owing to the unfavourable circumstances. Assume that the decrease occurs in proportion to the density and put the proportional constant k. Then

$$\frac{\partial S_2}{\partial t}=D\frac{\partial^2 S_2}{\partial x^2}-v\frac{\partial S_2}{\partial x}-kS_2. \quad \text{......(9)}$$

Substitution of
$$S_2=U\cdot e^{\frac{v}{2D}x-\frac{v^2t}{4D}-kt}, \quad \text{......(10)}$$

gives
$$\frac{\partial U}{\partial t}=D\frac{\partial^2 U}{\partial x^2}. \quad \text{......(11)}$$

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1) Frank, P.: Riemann-Wabers Pertiellen Differentialgleichungen der mathematischen Physik. 7. Aufl. Bd. II. p. 188.

2) For more general case, see l. c. p. 190.

As a simple case, taking U in the form of (5), we have

$$S_2 = \frac{1}{2\sqrt{\pi D t}} e^{-\frac{(x-vt)^2}{4Dt} - kt}. \qquad \text{(12)}$$

Putting $S_2 = \text{const.} = C$,

$$x - vt = \pm 2\sqrt{Dt\left\{\log\left(\frac{1}{2\sqrt{\pi D t C}}\right) - kt\right\}}. \qquad \text{(13)}$$

$(x-vt)^2$ attains its maximum, when

$$\sqrt{2\pi D t}\, C e^{2kt} = \frac{1}{\sqrt{e}}. \qquad \text{(14)}$$

II. TWO DIMENSIONAL DIFFUSION IN THE UNIFORM FLOW.

Consider that the density of the drifting bodies S_3 varies by two dimensional diffusion in the current, then

$$\frac{\partial S_3}{\partial t} = a_x^2 \frac{\partial^2 S_3}{\partial x^2} + a_y^2 \frac{\partial^2 S_3}{\partial y^2} - v \frac{\partial S_3}{\partial x}, \qquad \text{(15)}$$

where a_x and a_y are the coefficients of diffusion in the direction of x- and y-axes respectively, and t the time of travel.

Substituting $\dfrac{x}{a_x} = \xi$ and $\dfrac{y}{a_y} = \eta$, we get

$$\frac{\partial S_3}{\partial t} = \frac{\partial^2 S_3}{\partial \xi^2} + \frac{\partial^2 S_3}{\partial \eta^2} - \frac{v}{a_x} \frac{\partial S}{\partial \xi}, \qquad \text{(16)}$$

and putting $S_3 = S'' e^{\frac{v}{2a_x}\xi - \frac{v^2 t}{4a_x^2}}$, $\quad \dfrac{\partial S''}{\partial t} = \dfrac{\partial^2 S''}{\partial \xi^2} + \dfrac{\partial^2 S''}{\partial \eta^2}. \qquad \text{(17)}$

Further put $r = \sqrt{\xi^2 + \eta^2}$ and $S'' = T.R$, then

$$T = A e^{-\lambda^2 t}, \text{ and } \frac{\partial^2 R}{\partial \xi^2} + \frac{\partial^2 R}{\partial \eta^2} \equiv \frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} = -\lambda^2 R.$$

Put $\rho = \lambda r$, then $\quad \dfrac{d^2 R}{d\rho^2} + \dfrac{1}{\rho}\dfrac{dR}{d\rho} + R = 0$, and $R = J_0(\rho) = J_0(\lambda r)$.

Hence, $\quad S'' = \displaystyle\int_0^\infty A(\lambda) e^{-\lambda^2 t} J_0(\lambda r)\, d\lambda.$

Now let $S'' = \Phi(r)$ at $t = 0$, then, since

$$\Phi(r) = \int_0^\infty J_0(\lambda r)\lambda\, d\lambda \int_0^\infty \Phi(\alpha) J_0(\lambda\alpha)\alpha\, d\alpha,$$

$$S'' = \int_0^\infty \Phi(\alpha)\alpha\, d\alpha \int_0^\infty e^{-\lambda^2 t} J_0(\lambda r) J_0(\lambda\alpha)\lambda\, d\lambda. \qquad \text{(18)}$$

Next, let at $t = 0$, $\displaystyle\int_0^\varepsilon \alpha\Phi(\alpha)\, d\alpha = 1$ for $0 \leqq \alpha \leqq \varepsilon$, and $\Phi(\alpha) = 0$ for $\alpha > \varepsilon$, ε

being vanishingly small, ...(19)[1)]

then $$S_3=\frac{1}{2t}e^{-\frac{\frac{x^2}{a_x^2}+\frac{y^2}{a_y^2}}{4t}-\frac{v^2t}{4a_x^2}+\frac{v}{2a_x^2}x}=\frac{1}{2t}e^{-\frac{\frac{(x-vt)^2}{a_x^2}+\frac{y^2}{a_y^2}}{4t}}. \qquad \ldots\ldots(20)$$

The isoline $S_3=\text{const.}=C$ represents the following ellipse :

$$\frac{(x-vt)^2}{a_x^2}+\frac{y^2}{a_y^2}=\log\left(\frac{1}{2tC}\right)^{4t}. \qquad \ldots\ldots(21)$$

The centre of this ellipse progresses in the direction of x with the velocity v as time elapses. The axes of the ellipse are

$$a=2a_x\sqrt{t}\sqrt{\log(2tC)^{-1}} \text{ and } b=2a_y\sqrt{t}\sqrt{\log(2tC)^{-1}}. \qquad \ldots\ldots(22)$$

The ellipse exists when $\frac{1}{2t}\geqq C$. a attains its maximum when

$$\frac{\partial a}{\partial t}=a_x\frac{\log(2tC)^{-1}-1}{\sqrt{t\log(2tC)^{-1}}}=0, \quad \text{or} \quad \log(2Ct)=-1, \quad \text{or} \quad 2Ct=0.368 \ldots(23)$$

The extremum value of a is $2a_x\sqrt{\frac{0.368}{2C}}=0.858\frac{a_x}{\sqrt{C}}$. ....(24)

Again, when $a=\text{const.}$,

$$t\log\left(\frac{1}{2tC}\right)=\frac{a^2}{4a_x^2}=\text{const}, \quad \text{or} \quad C=\frac{1}{2t}e^{-\frac{a^2}{4a_x^2t}}. \qquad \ldots\ldots(25)$$

The time when C attains its maximum is given by

$$\frac{\partial C}{\partial t}=-\frac{1}{2t^2}e^{-\frac{a^2}{4a_x^2}}+\frac{a^2}{8t^3a_x^2}e^{-\frac{a^2}{4a_x^2t}}=0, \qquad \text{i.e.} \quad t=\frac{1}{4}\left(\frac{a}{a_x}\right)^2. \qquad \ldots\ldots(26)$$

The extremum value of C is $\frac{2a_x^2}{a^2}e^{-1}=0.736\left(\frac{a_x}{a}\right)$. ....(27)

The above results may be applied to the problems of current-bottle experiments, as well as of moving salty water-masses in the Tusima Current System and of the diffusion of polluted water-masses in the river.

DISCUSSIONS.

For example, taking the case of current-bottle experiments referred to previously (Fig. 1), we can check the formulae in the paragraphs I and II. Now let us consider the scattering of bottles as two-dimensional phenomena. The range of the distribution of the recovered bottles U (sea-miles) in Fig. 1 will be proportional to the size of ellipse. Then using the formula (22) in II, we have $U\propto\sqrt{t}\sqrt{\log(2tC)^{-1}}$....(28). Plotting the curves computed, with

1) In this case, total number of bottles $N=2\pi\int_0^{\varepsilon}\Phi(\alpha)\alpha d\alpha=2\pi$.

the values of C, $\frac{1}{1000}$, $\frac{1}{2000}$ and $\frac{1}{3000}$ respectively, (Fig. 2), we see that this relation is approximately established. Next, we consider the case as one-dimensional phenomena. Then, from the formulae (7) and (13) in I, we may put $U \propto \sqrt{t}\sqrt{\log\left(\frac{C'}{\sqrt{t}}\right)}$ ....(29). Plotting the curve computed when $C'=100$ (Fig. 2), we see again more or less fitted to the observed relation.

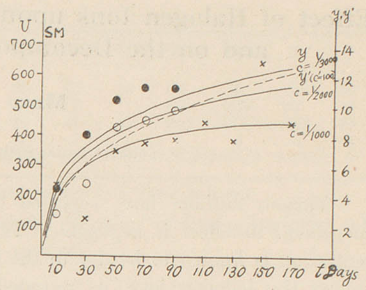

Fig. 2. ●Bottles thrown in Wakasa Bay (July, 1930); × Bottles thrown in Tusima Channel (June, 1932); ○Bottles thrown in Tusima Channel (Oct., 1933).
Full lines: $y=\sqrt{t \log (2tC)^{-1}}$;
Broken line: $y'=\sqrt{t \log (C'/\sqrt{t})}$.

In the above treatment we have neglected the effect of topographical conditions on the recovery of current-bottles, i.e. such as the capture of the bottles by the vortical currents or the accumulation of them on converging-line of the current about penninsulas, etc. But in actual cases, as we have already noticed,[1,2] in some regions such as Toyama Bay, Wakasa Bay, Noto- and Oga-Pennisula, Tugaru Strait, etc., the bottles are especially much in number recovered. This fact will make it difficult to determine the suitable value of U in some cases.

Lastly it may be remarked that O. G. SUTTON's formula[3] on the scattering of small balloons, which was proved suitable for the three dimensional diffusion of the atmosphere according to the experimental study of Messrs. T. SAITO and M. YOSITAKE,[4] shows some difference from our results i.e., $\sigma^2=\frac{1}{2}c^2(ut)^m$, where σ denotes the standard deviation of the balloons from their mean path, c, m are two constants, u is the mean wind velocity and t is the time of travel.

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1) M UDA: Hydrographical investigations in the seas adjacent to Wakasa Bay. Rec. Oceanogr. Works in Japan, 4 (1), p. 24, 25, (1932).

2) M. UDA: Hydrographical studies based on simultaneous oceanographical surveys made in the Japan Sea and in its adjacent waters during May and June, 1932. Rec. Oceanogr. Works in Japan. 4 (1), pp. 82~85, (1934).

3) Proc. Roy. Soc., A. 135. p. 143, (1932).

4) Jour. Meteor. Soc. of Japan, 12 (9), pp. 469~471, (1934).